

# A Note On The Spectral Norms of The Matrices Connected Integer Numbers Sequence

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## Abstract

In this paper, we compute the spectral norms of the matrices related with integer sequences and we give two examples related with Fibonacci and Lucas numbers.

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## 1 Introduction

In [1], the upper and lower bounds for the spectral norms of  $r$ -circulant matrices are obtained by Shen and Cen. The lower bounds for the norms of Cauchy-Toeplitz and Cauchy-Hankel matrices are given by Wu in [2]. In [3-5], Solak and Bozkurt have found some bounds for the norms of Cauchy-Toeplitz, Cauchy-Hankel and circulant matrices. In [5], Barrett and Feinsilver have defined the principal 2- minors and have formulated the inverse of tridiagonal matrix.

Let  $A$  be any  $n \times n$  complex matrix. The well known spectral norm of the matrix  $A$  is

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} |\lambda_i(A^H A)|}$$

where  $\lambda_i(A^H A)$  is eigenvalue of  $A^H A$  and  $A^H$  is conjugate transpose of the matrix  $A$ .  $k$ -principal minor of the matrix  $A$  is denoted by

$$A \left( \begin{array}{c} i_1 i_2 \dots i_k \\ i_1 i_2 \dots i_k \end{array} \right) = \begin{vmatrix} a_{i_1, i_1} & a_{i_1, i_2} & \dots & a_{i_1, i_k} \\ a_{i_2, i_1} & a_{i_2, i_2} & \dots & a_{i_2, i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k, i_1} & a_{i_k, i_2} & \dots & a_{i_k, i_k} \end{vmatrix} \quad (1)$$

where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  ( $1 \leq k \leq n$ )[5].

Now we define our matrix.  $x_i$ s are any integer numbers sequence for  $i = 1, 2, \dots, n$ . Let matrix  $A_x$  be following form:

$$A_x = [a_{ij}]_{i,j=1}^n = [x_i - x_j]_{i,j=1}^n \quad (2)$$

Obviously,  $A_x$  is skew-symmetric matrix. i.e.  $A_x^T = -A_x$ . Since eigenvalues of the skew-hermitian matrix  $A_x$  are pure imaginary, eigenvalues of the matrix  $iA_x$  are real where  $i$  is complex unity.

## 2 Main Result

**Theorem 1** *Let the matrices  $A_x$  be as in (1). Then*

$$\|A_x\|_2 = \sum_{1 \leq r < s \leq n} (x_r - x_s)^2. \quad (3)$$

where  $n \geq 4$ .

**Proof.** If we subtract  $(i-1)$ th row from  $i$ th row of the matrix  $A_x$  for  $i = n, n-1, \dots, 2$ , then we obtain

$$B_x = \begin{bmatrix} 0 & x_1 - x_2 & x_1 - x_3 & \dots & x_1 - x_n \\ x_2 - x_1 & x_2 - x_1 & x_2 - x_3 & \dots & x_2 - x_n \\ x_3 - x_2 & x_3 - x_2 & x_3 - x_3 & \dots & x_3 - x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} - x_{n-2} & x_{n-1} - x_{n-2} & x_{n-1} - x_{n-3} & \dots & x_{n-1} - x_n \\ x_n - x_{n-1} & x_n - x_{n-1} & x_n - x_{n-2} & \dots & x_n - x_n \end{bmatrix}.$$

Obviously,  $\text{rank}(B_x) = \text{rank}(A_x) = 2$ . Since the matrix  $A_x$  is skew-symmetric, the matrix  $iA_x$  is symmetric where  $i$  is complex unity. Then all the eigenvalues of the matrix  $iA_x$  are real numbers. Moreover,  $\text{rank}(A_x) = \text{rank}(iA_x)$ . Since determinants of all  $k$ -square submatrices of the matrix  $iA_x$  are zero for  $k \geq 3$ , all principal  $k$ -minors of the matrix  $iA_x$  are zero for  $k \geq 3$ . Then characteristic polynomial of the matrix  $iA_x$

$$\Delta_{iA_x}(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2}$$

where  $(-1)^k a_k$  is the sum of principal  $k$ -minors of the matrix  $iA_x$  where  $k = 1, 2$ . On the other hand since  $\text{rank}(iA_x) = 2$ , two eigenvalues of the matrix  $iA_x$  are nonzero. If  $i\lambda$  is the eigenvalue of the matrix  $A_x$ , then  $-i\lambda$  is an eigenvalue of  $A_x$ . Then  $a_1 = -\text{tr}(A_x) = -\text{tr}(iA_x) = \sum_{k=1}^n \lambda_k = 0$  where  $\lambda_k$  are the eigenvalues of the matrix  $iA_x$ . Coefficient  $a_2$  is the sum of principal 2-minors of any square matrix  $A_x$ . i.e.

$$a_2 = \sum_{1 \leq r < s \leq n} A \begin{pmatrix} r & s \\ r & s \end{pmatrix}.$$

Then we have

$$\begin{aligned} a_2 &= \sum_{1 \leq r < s \leq n} iA_x \begin{pmatrix} r & s \\ r & s \end{pmatrix} = \sum_{1 \leq r < s \leq n} \begin{vmatrix} i(x_r - x_r) & i(x_r - x_s) \\ i(x_s - x_r) & i(x_s - x_s) \end{vmatrix} \\ &= \sum_{1 \leq r < s \leq n} \begin{vmatrix} 0 & i(x_r - x_s) \\ -i(x_r - x_s) & 0 \end{vmatrix} = - \sum_{1 \leq r < s \leq n} (x_r - x_s)^2. \end{aligned}$$

Hence we obtain

$$\Delta_{iA_x}(\lambda) = \lambda^n - \left( \sum_{1 \leq r < s \leq n} (x_r - x_s)^2 \right) \lambda^{n-2}.$$

Then

$$\|iA_x\|_2 = \|A_x\|_2 = \sum_{1 \leq r < s \leq n} (x_r - x_s)^2.$$

The proof is completed. ■

### 3 Numerical Examples

The well known  $F_n$  is  $n$ th Fibonacci number with recurrence relation  $F_n = F_{n-1} + F_{n-2}$  initial condition  $F_0 = 0$  and  $F_1 = 1$  and  $L_n$  is  $n$ th Lucas number with recurrence relation  $L_n = L_{n-1} + L_{n-2}$  initial condition  $L_0 = 2$  and  $L_1 = 1$ . The matrices  $F$  and  $L$  are following forms:

$$F = [F_i - F_j]_{i,j=1}^n$$

and

$$L = [L_i - L_j]_{i,j=1}^n.$$

Let  $\alpha = \frac{1 + \sqrt{5}}{2}$  and  $\beta = \frac{1 - \sqrt{5}}{2}$ . Then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n$$

where  $F_n$  and  $L_n$  are  $n$ th Fibonacci and Lucas numbers, respectively. Furthermore

$$L_n = F_{n-1} + F_{n+1}.$$

Now, we compute the spectral norms of the matrices  $F$  and  $L$ . By the definition of the spectral norm, we have

$$\|F\|_2 = \sum_{1 \leq r < s \leq n} (F_r - F_s)^2 = (n-1)F_{n+1}F_n - 2 \sum_{r=1}^{n-1} \sum_{s=r+1}^n F_r F_s.$$

By the relationship between Fibonacci and Lucas numbers, we obtain

$$\|F\|_2 = \begin{cases} (n-1)F_{n+1}F_n - \frac{2}{5} \left( L_n - 2 + \sum_{r=1}^{n-1} \sum_{s=r+1}^n L_{r+s} \right), & n \text{ is even} \\ (n-1)F_{n+1}F_n - \frac{2}{5} \left( L_n - 1 + \sum_{r=1}^{n-1} \sum_{s=r+1}^n L_{r+s} \right), & n \text{ is odd} \end{cases}.$$

Similarly,

$$\|L\|_2 = \sum_{1 \leq r < s \leq n} (L_r - L_s)^2 = (n-1)(L_{n+1}L_n - 2) - 2 \sum_{r=1}^{n-1} \sum_{s=r+1}^n L_r L_s.$$

Then

$$\|L\|_2 = \begin{cases} (n-1)(L_{n+1}L_n - 2) - 2 \left( L_n - 2 + \sum_{r=1}^{n-1} \sum_{s=r+1}^n L_{r+s} \right), & n \text{ is even} \\ (n-1)(L_{n+1}L_n - 2) - 2 \left( L_n - 1 + \sum_{r=1}^{n-1} \sum_{s=r+1}^n L_{r+s} \right), & n \text{ is odd} \end{cases}.$$

## References

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